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A formula for the generalized twist

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Abstract

In this paper, a new formula for the generalized twist is presented which is valid for linearly elastic, nonhomogeneous and anisotropic beams of solid cross section. The generalized twist is expressed in terms of axial component of the infinitesimal rotation vector weighted by the stress function of Saint-Venant's torsional problem. Characterization of a torsion-free bending problem is also presented by the use of the generalized twist.

Keywords: Anisotropic beam; Generalized twist; Torsion; Shear centre

1. Introduction

Consider a beam of solid cross section bounded by a cylindrical surface ("side-surface") and two planes ("end cross sections") normal to the side surface. It is assumed that, the body forces are absent, that the side surface of the beam is free from external stresses and that the given forces satisfying the equilibrium conditions of the body as whole are shearing stresses applied to the end cross sections of the beam.

Three-dimensional rectangular Cartesian coordinate system $(O; x, y, z)$ will be used. The unit vectors of the coordinate system $(O; x, y, z)$ are $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. The Oz axis is directed parallel to the generators of the side surface and the plane Oxy chosen to coincide with the "lower" of the ends of the beam. The upper end of the beam will then have the coordinate $z = L$, where L is the length of the beam (Fig. 1).

The material of the beam is linearly elastic, nonhomogeneous and anisotropic. The material properties of the cylindrical beam do not depend on the axial coordinate z. The assumed form of anisotropy is described by the equations (Lekhnitskii, 1963; Milne-Thomson, 1962)

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy}, \tag{1.1}$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy}, \tag{1.2}$$

$$\varepsilon_z = a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy}, \tag{1.3}$$

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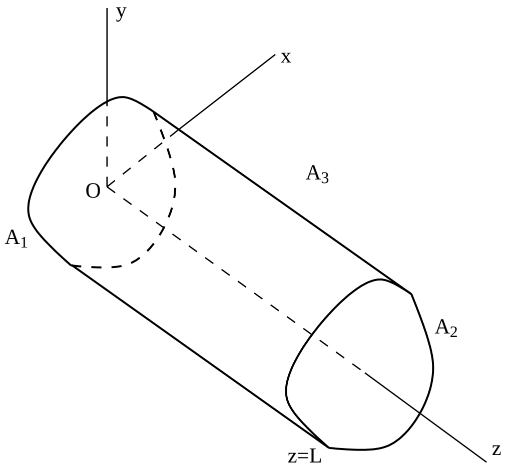

Fig. 1. Elastic beam of solid cross section.

$$\gamma_{yz} = a_{44}\tau_{yz} + a_{45}\tau_{xz}, \tag{1.4}$$

$$\gamma_{xz} = a_{45}\tau_{yz} + a_{55}\tau_{xz}, \tag{1.5}$$

$$\gamma_{xy} = a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy}. \tag{1.6}$$

According to the nonhomogenity which appears in the plane of cross section the coefficients a_{ij} may depend only on the cross-sectional coordinates x and y. In Eq. (1) ε_x, ε_y, ε_z are direct strains, γ_{xy}, γ_{xz}, γ_{yz} are shearing strains, σ_x, σ_y, σ_z are normal stresses and τ_{xy}, τ_{xz}, τ_{yz} are shearing stresses. From Eqs. (1.1)–(1.6) can be read out that there exists a plane of symmetry of elasticity. The plane of symmetry of elasticity is the plane Oxy (Lekhnitskii, 1963; Milne-Thomson, 1962).

It was mentioned the cylindrical beam is loaded by tangential surface forces on its end cross sections only. We denote by A_1 and A_2 the end cross sections at $z = 0$ and at $z = L$, respectively. The surface tractions on A_1 and A_2 are specified as

$$\mathbf{p}_1 = X_1(x,y)\mathbf{e}_x + Y_1(x,y)\mathbf{e}_y \quad \text{on } A_1, \tag{2.1}$$

$$\mathbf{p}_2 = X_2(x,y)\mathbf{e}_x + Y_2(x,y)\mathbf{e}_y \quad \text{on } A_2. \tag{2.2}$$

From the conditions of equilibrium it follows that

$$\int_{A_i} X_i(x,y)\,\mathrm{d}A = \int_{A_i} Y_i(x,y)\,\mathrm{d}A = 0 \quad (i = 1,2), \tag{3}$$

$$T = \int_{A_2} (xY_2 - yX_2)\,\mathrm{d}A = -\int_{A_1} (xY_1 - yX_1)\,\mathrm{d}A. \tag{4}$$

In Eq. (4) T is the applied torque, its value is given.

Let $\mathbf{u}$ be a sufficiently smooth equilibrium displacement field. The associated strain and stress fields with $\mathbf{u}$ are denoted by $\mathbf{E}(\mathbf{u})$ and $\mathbf{S}(\mathbf{u})$. The connection between the stress field $\mathbf{S}(\mathbf{u})$ and strain field $\mathbf{E}(\mathbf{u})$ is formulated in Eq. (1) and the strain field is derived from $\mathbf{u}$ by the equation $\mathbf{E}(\mathbf{u}) = \mathrm{def}\,\mathbf{u}$ (Lurje, 1970).

The domain occupied by the cylindrical beam is a three-dimensional space domain V with boundary surface ∂V. The boundary surface ∂V is divided into three parts as $\partial V = A_1 \cup A_2 \cup A_3$. Here, A_3 is the cylindrical surface segment of ∂V. The surface traction associated with the stress tensor is

$$\mathbf{s}(\mathbf{u}) = \mathbf{S}(\mathbf{u})\mathbf{n}, \tag{5}$$

where $\mathbf{n}$ is the outward unit normal to ∂V.

The equilibrium displacement field $\mathbf{u}$ is a solution of the generalized Saint-Venant problem of torsion if all the field equations of elasticity (strain–displacement relations, Hooke's law formulated in Eq. (1) and the equations of equilibrium with zero body forces) are satisfied under the next boundary conditions

$$\mathbf{s}(\mathbf{u}) = \mathbf{0} \quad \text{on } A_3, \quad \mathbf{s}(\mathbf{u}) = \mathbf{p}_1 \quad \text{on } A_1, \quad \mathbf{s}(\mathbf{u}) = \mathbf{p}_2 \quad \text{on } A_2. \tag{6}$$

We note here, $\mathbf{p}_1$ and $\mathbf{p}_2$ have the form in accordance with Eq. (2) and their components X_i, Y_i $(i = 1, 2)$ satisfy *only* the conditions (3) and (4), where T is a given value. This means that the different tangential surface forces on the end cross sections can define a generalized Saint-Venant problem of torsion.

One of the solutions of the generalized Saint-Venant problem of torsion is

$$\mathbf{u}_s = -\vartheta yz\mathbf{e}_x + \vartheta xz\mathbf{e}_y + \vartheta\varphi(x,y)\mathbf{e}_z, \tag{7}$$

where the function $\varphi = \varphi(x,y)$ is a solution to the boundary value problem (Lekhnitskii, 1963, 1971; Lomakin, 1976)

$$\frac{\partial}{\partial x}\left(A_{45}\left(\frac{\partial\varphi}{\partial y} + x\right) + A_{55}\left(\frac{\partial\varphi}{\partial x} - y\right)\right) + \frac{\partial}{\partial y}\left(A_{44}\left(\frac{\partial\varphi}{\partial y} + x\right) + A_{45}\left(\frac{\partial\varphi}{\partial x} - y\right)\right) = 0 \quad \text{in } A, \tag{8}$$

$$\left(A_{45}\left(\frac{\partial\varphi}{\partial y} + x\right) + A_{55}\left(\frac{\partial\varphi}{\partial x} - y\right)\right)n_x + \left(A_{44}\left(\frac{\partial\varphi}{\partial y} + x\right) + A_{45}\left(\frac{\partial\varphi}{\partial x} - y\right)\right)n_y = 0 \quad \text{on } \partial A. \tag{9}$$

Here, A is the cross section of the beam, it is a simply connected bounded plane domain, ∂A is the boundary curve of A, n_x, n_y are the components of unit outward normal vector to curve ∂A, and

$$A_{44} = \frac{a_{55}}{\triangle}, \quad A_{45} = -\frac{a_{45}}{\triangle}, \quad A_{55} = \frac{a_{44}}{\triangle}, \quad \triangle = a_{44}a_{55} - a_{45}^2.$$

In the expression of $\mathbf{u}_s$ ϑ denotes the twist.

The stress field associated with the displacement field $\mathbf{u}_s$ is as follows

$$\sigma_x^s = \sigma_y^s = \sigma_z^s = \tau_{xy}^s = 0 \quad \text{in } V \cup \partial V, \tag{10}$$

$$\tau_{xz}^s = \vartheta\frac{\partial U}{\partial y}, \quad \tau_{yz}^s = -\vartheta\frac{\partial U}{\partial x} \quad \text{in } V \cup \partial V, \tag{11}$$

and the stress function $U = U(x,y)$ is the solution to the boundary value problem (Lekhnitskii, 1971; Lomakin, 1976)

$$\frac{\partial}{\partial x}\left(a_{44}\frac{\partial U}{\partial x} - a_{45}\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial y}\left(-a_{45}\frac{\partial U}{\partial x} + a_{55}\frac{\partial U}{\partial y}\right) = -2 \quad \text{in } A, \tag{12}$$

$$U = 0 \quad \text{on } \partial A. \tag{13}$$

The connection between the warping function $\varphi = \varphi(x,y)$ and the stress function $U = U(x,y)$ is formulated in equations (Lekhnitskii, 1971; Lomakin, 1976)

$$\frac{\partial \varphi}{\partial x} - y = -a_{45}\frac{\partial U}{\partial x} + a_{55}\frac{\partial U}{\partial y}, \tag{14}$$

$$\frac{\partial \varphi}{\partial y} + x = -a_{44}\frac{\partial U}{\partial x} + a_{45}\frac{\partial U}{\partial y}. \tag{15}$$

Torque–twist relationship is

$$T = S\vartheta, \tag{16}$$

where S is the torsional rigidity of the cross section (Lekhnitskii, 1971; Lomakin, 1976)

$$S = 2\int_A U \, \mathrm{d}A. \tag{17}$$

The solution above detailed was offered by Saint-Venant for homogeneous beam in his celebrated memoir (de Saint-Venant, 1856). Saint-Venant's solution of pure torsion is based on the warping function $\varphi = \varphi(x, y)$. Prandtl gave the solution of the uniform (pure) torsion problem for homogeneous isotropic beam by the use of stress function (Prandtl, 1904).

In connection with the generalized Saint-Venant's problem of torsion Truesdell (Truesdell, 1959, 1966, 1978) has proposed the following problem.

For an isotropic, linearly elastic cylinder subject to end tangential tractions equipollent to a torque T, define the "twist" τ in such a way that

$$T = (\mu R)\tau(\mathbf{u}), \tag{18}$$

where $S = \mu R$, the torsional rigidity of the cylinder, is resolved into the factors μ, the shear modulus, and R, a geometric quantity (Saint-Venant's torsional constant) depending only on the cross section.

The designation $\tau(\mathbf{u})$ stresses that the generalized twist depends on the considered solution of the generalized torsional problem. Truesdell remarked that $\tau(\mathbf{u})$ would generalize Saint-Venant's notion of twist so as to apply also to solutions of the torsional problem corresponding to distributions of end tractions different from the one assumed by Saint-Venant (Ieaşan, 1986, 1987).

Day and Podio-Guiduglio (Day, 1981; Podio Guiduglio, 1983) solved the Truesdell's problem for homogeneous isotropic beam.

The solution and generalization of Truesdell's problem for nonhomogeneous anisotropic beam was presented by Ieaşan. Ieaşan considered the case of coupled torsion-bending-tension problem of anisotropic beams (Ieaşan, 1986, 1987).

The aim of the present paper is to derive a new formula for the generalized twist if the anisotropy of nonhomogeneous elastic beam is specified by Eq. (1). Expression of the generalized twist will be given in terms of Prandtl's stress function of the uniform torsion and the axial component of the infinitesimal rotation vector of the generalized Saint-Venant's torsional problem.

2. Generalized twist

Let $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ be a solution of the generalized torsional problem defined by prescriptions (2)–(4) and (6). The axial component of the infinitesimal rotation vector can be computed as (Lurje, 1970; Sokolnikoff, 1956):

$$\omega_z(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (x, y, z) \in V \cup \partial V. \tag{19}$$

We introduce the U-weighted mean value of $\omega_z(\mathbf{u})$ which relates to the cross sections, by the definition

$$\Omega(z) = \frac{\int_A U(x,y)\omega_z(x,y,z)\,\mathrm{d}x\,\mathrm{d}y}{\int_A U(x,y)\,\mathrm{d}x\,\mathrm{d}y} \quad 0 \leqslant z \leqslant L. \tag{20}$$

It is evident, if $\mathbf{u} = \mathbf{u}_s$ then we have $\Omega(z) = \vartheta z$.

Theorem 1. *Let* $\mathbf{u}$ *be an arbitrary solution of the generalized torsional problem for a given value of torque* T. *The relationship*

$$T = S\tau(\mathbf{u}) \tag{21}$$

is valid. Here $\tau(\mathbf{u})$ *is the generalized twist defined by the formula*

$$\tau(\mathbf{u}) = \frac{\Omega(L) - \Omega(0)}{L}. \tag{22}$$

Proof. The proof of the statement formulated in formula (21) is based on Betti's theorem (Lurje, 1970; Sokolnikoff, 1956) which says that in this case

$$\int_{\partial V} \mathbf{s}(\mathbf{u}) \cdot \mathbf{u}_s\,\mathrm{d}A = \int_{\partial V} \mathbf{s}(\mathbf{u}_s) \cdot \mathbf{u}\,\mathrm{d}A. \tag{23}$$

Here, the dot between two vectors denotes their scalar product and ϑ in $\mathbf{u}_s$ is chosen to be one. We have

$$\int_{\partial V} \mathbf{s}(\mathbf{u}) \cdot \mathbf{u}_s\,\mathrm{d}A = \int_{A_2} \mathbf{s}(\mathbf{u}) \cdot \mathbf{u}_s\,\mathrm{d}A = L\int_{A_2} (xY_2 - yX_2)\,\mathrm{d}A = TL, \tag{24}$$

according to Eqs. (2.2), (4) and (7). On the other hand, we can write

$$\begin{aligned}
\int_{\partial V} \mathbf{s}(\mathbf{u}_s) \cdot \mathbf{u}\,\mathrm{d}A &= \int_{A_1} \mathbf{s}(\mathbf{u}_s) \cdot \mathbf{u}\,\mathrm{d}A + \int_{A_2} \mathbf{s}(\mathbf{u}_s) \cdot \mathbf{u}\,\mathrm{d}A \\
&= -\int_{A_1} \left(u(x,y,0)\frac{\partial U}{\partial y} - v(x,y,0)\frac{\partial U}{\partial x} \right)\mathrm{d}A + \int_{A_2} \left(u(x,y,L)\frac{\partial U}{\partial y} - v(x,y,L)\frac{\partial U}{\partial x} \right)\mathrm{d}A \\
&= -\int_{\partial A_1} (u(x,y,0)n_y - v(x,y,0)n_x)U\,\mathrm{d}s - \int_{A_1} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_{z=0} U\,\mathrm{d}A \\
&\quad + \int_{\partial A_2} (u(x,y,L)n_y - v(x,y,L)n_x)U\,\mathrm{d}s + \int_{A_2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_{z=L} U\,\mathrm{d}A \\
&= 2\left\{ \int_{A_2} \omega_z(x,y,L)U(x,y)\,\mathrm{d}A - \int_{A_1} \omega_z(x,y,0)U(x,y)\,\mathrm{d}A \right\} = S(\Omega(L) - \Omega(0)).
\end{aligned} \tag{25}$$

Here, we have integrated by parts two times and Stokes theorem, Eqs. (11), (13), (17), (19), (20) have been used. In Eq. (25), the arc coordinate defined on boundary curve ∂A_i $(i = 1, 2)$ has been denoted by s.

Eq. (24) was derived by Day and Podio-Guidugli (Day, 1981; Podio Guiduglio, 1983).

Substitution of Eqs. (24) and (25) into Eq. (23) leads to the formula (21).

We note that the boundary conditions

$$u(x,y,0) = u_1(x,y), \quad v(x,y,0) = v_1(x,y), \quad \sigma_z(x,y,0) = 0, \tag{26}$$

$$u(x,y,L) = u_2(x,y), \quad v(x,y,L) = v_2(x,y), \quad \sigma_z(x,y,L) = 0, \tag{27}$$

$$\mathbf{s}(\mathbf{u}) = \mathbf{0} \quad \text{on } A_3 \tag{28}$$

determine a mixed type 3D elastostatics boundary value problem of the anisotropic nonhomogeneous beam. For given values of u_1, v_1 and u_2, v_2 this boundary value problem has a unique solution. From the conditions of equilibrium is follows that the section forces

$$N = \int_A \sigma_z(x,y,z)\,\mathrm{d}A, \quad V_x = \int_A \tau_{xz}(x,y,z)\,\mathrm{d}A, \quad V_y = \int_A \tau_{yz}(x,y,z)\,\mathrm{d}A \tag{29}$$

and section moments

$$M_x = \int_A y\sigma_z(x,y,z)\,\mathrm{d}A, \quad M_y = -\int_A x\sigma_z(x,y,z)\,\mathrm{d}A \tag{30}$$

vanish and the torque

$$T = \int_A (x\tau_{yz}(x,y,z) - y\tau_{xz}(x,y,z))\,\mathrm{d}A \tag{31}$$

does not depend on the axial coordinate z.

This mixed type 3D boundary value problem is a generalized torsional problem specified by the given surface displacements u_1, v_1 and u_2, v_2. By the application of formula (21) we get the value of the torque T transmitted by the beam in the terms u_1, v_1 and u_2, v_2 without knowing the solution of the corresponding mixed type 3D elastostatics boundary value problem.

3. A characterization of the torsion-free bending

Let the anisotropic inhomogeneous elastic beam shown in Fig. 1 be loaded by tangential surface forces on the end cross section A_2. The resultant of the system of tangential surface forces is $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$. Let the other end cross section of the beam be fixed. It is assumed that the body forces and the surface forces on A_3 vanish. According to above formulated prescriptions we have

$$u(x,y,0) = v(x,y,0) = w(x,y,0) = 0, \tag{32}$$

$$\tau_{xz}(x,y,L) = X_2(x,y), \quad \tau_{yz}(x,y,L) = Y_2(x,y), \quad \sigma_z(x,y,L) = 0, \tag{33}$$

$$\mathbf{s}(\mathbf{u}) = \mathbf{0} \quad \text{on } A_3, \tag{34}$$

and

$$P = \int_{A_2} X_2(x,y)\,\mathrm{d}A, \quad Q = \int_{A_2} Y_2(x,y)\,\mathrm{d}A, \quad P^2 + Q^2 \neq 0. \tag{35}$$

This type of loading conditions is used to model the shear behavior of anisotropic inhomogeneous elastic beam (Lekhnitskii, 1963; Sarkisjan, 1970).

We note here, if the Saint-Venant's flexure solution is used as a solution of tip loaded beam then the displacement boundary conditions can be satisfied only weak sense such as

$$u = v = w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{at } x = y = z = 0 \tag{36}$$

and the stress field including the distributed surface forces on A_1 and A_2 follows the Saint-Venant's flexure solution (Lekhnitskii, 1963; Sarkisjan, 1970).

According to Trefftz (1935) in the state of torsion-free bending caused by a tip load there is no interaction between the fields of pure torsion and the fields of flexure. This means that the strain energy in pure torsion and in bending without torsion should be uncoupled. Missing of interaction between the pure torsion and the bending caused by tip loads is characterized by the equation

$$W_{FT} = \int_0^L \left(\int_A \left(\gamma_{xz}^F \tau_{xz}^T + \gamma_{yz}^F \tau_{yz}^T \right) \mathrm{d}A \right) \mathrm{d}z = 0, \tag{37}$$

where W_{FT} is the mixed strain energy computed on the equilibrium states of pure torsion (with the unit value of ϑ) and bending without torsion. Here, γ_{xz}^F and γ_{yz}^F are derived from the displacement field of flexure solution as

$$\gamma_{xz}^F = \frac{\partial u^F}{\partial z} + \frac{\partial w^F}{\partial x}, \quad \gamma_{yz}^F = \frac{\partial v^F}{\partial z} + \frac{\partial w^F}{\partial y} \tag{38}$$

and τ_{xz}^T, τ_{yz}^T can be obtained from Eq. (11) with $\vartheta = 1$.

For solid cross section from Eq. (37) we get

$$\begin{aligned} W_{FT} &= \int_0^L \left(\int_A \left(\gamma_{xz}^F \frac{\partial U}{\partial y} - \gamma_{yz}^F \frac{\partial U}{\partial x} \right) \mathrm{d}A \right) \mathrm{d}z \\ &= \int_0^L \left(\int_{\partial A} (\gamma_{xz}^F n_y - \gamma_{yz}^F n_x) U \, \mathrm{d}s \right) \mathrm{d}z + \int_0^L \left(\int_A \left(\frac{\partial \gamma_{yz}^F}{\partial x} - \frac{\partial \gamma_{xz}^F}{\partial y} \right) U \, \mathrm{d}A \right) \mathrm{d}z \\ &= 2 \int_0^L \left(\int_A U(x,y) \frac{\partial \omega_z^F}{\partial z} \mathrm{d}A \right) \mathrm{d}z = 0. \end{aligned} \tag{39}$$

Here, we have integrated by parts and Stokes theorem, boundary condition (13) and the undermentioned equation (Lurje, 1970)

$$\frac{\partial \gamma_{yz}^F}{\partial x} - \frac{\partial \gamma_{xz}^F}{\partial y} = \frac{\partial^2 v^F}{\partial x \partial z} + \frac{\partial^2 w^F}{\partial x \partial y} - \frac{\partial^2 u^F}{\partial y \partial z} - \frac{\partial^2 w^F}{\partial y \partial x} = \frac{\partial}{\partial z} \left(\frac{\partial v^F}{\partial x} - \frac{\partial u^F}{\partial y} \right) = 2 \frac{\partial \omega_z^F}{\partial z} \tag{40}$$

have been used to derive Eq. (39). From Eq. (39) and the definition of generalized twist it follows that

$$\begin{aligned} W_{FT} &= 2 \left(\int_A \omega_z^F(x,y,L) U(x,y) \, \mathrm{d}A - \int_A \omega_z^F(x,y,0) U(x,y) \, \mathrm{d}A \right) \\ &= 2 \int_A U(x,y) \, \mathrm{d}A \left\{ \frac{\int_A \omega_z^F(x,y,L) U(x,y) \, \mathrm{d}A}{\int_A U(x,y) \, \mathrm{d}A} - \frac{\int_A \omega_z^F(x,y,0) U(x,y) \, \mathrm{d}A}{\int_A U(x,y) \, \mathrm{d}A} \right\} = S \frac{\Omega^F(L) - \Omega^F(0)}{L} L \\ &= SL\tau(\mathbf{u}^F) = 0. \end{aligned} \tag{41}$$

The validity of Eq. (41) can also be proven by means of Betti's reciprocal theorem (Lurje, 1970; Sokolnikoff, 1956). According to Betti's reciprocal theorem we can write

$$W_{FT} = \int_{\partial V} \mathbf{u}^F \cdot \mathbf{s}(\mathbf{u}^T) \, \mathrm{d}A. \tag{42}$$

Detailed computations such as used to derive Eqs. (39) and (40) yield

$$\begin{aligned}\int_{\partial V} \mathbf{u}^F \cdot \mathbf{s}(\mathbf{u}^T)\,\mathrm{d}A &= \int_{A_1} \mathbf{u}^F \cdot \mathbf{s}(\mathbf{u}^T)\,\mathrm{d}A + \int_{A_2} \mathbf{u}^F \cdot \mathbf{s}(\mathbf{u}^T)\,\mathrm{d}A \\ &= -\int_{A_1}\left(u^F(x,y,0)\frac{\partial U}{\partial y} - v^F(x,y,0)\frac{\partial U}{\partial x}\right)\mathrm{d}A + \int_{A_2}\left(u^F(x,y,L)\frac{\partial U}{\partial y} - v^F(x,y,L)\frac{\partial U}{\partial x}\right)\mathrm{d}A \\ &= -\int_{\partial A_1}(u^F(x,y,0)n_y - v^F(x,y,0)n_x)U\,\mathrm{d}s - \int_{A_1} U(x,y)\left(\frac{\partial v^F}{\partial x} - \frac{\partial u^F}{\partial y}\right)_{z=0}\mathrm{d}A \\ &\quad + \int_{\partial A_2}(u^F(x,y,L)n_y - v^F(x,y,L)n_x)U\,\mathrm{d}s + \int_{A_2} U(x,y)\left(\frac{\partial v^F}{\partial x} - \frac{\partial u^F}{\partial y}\right)_{z=L}\mathrm{d}A \\ &= 2\left(\int_{A_2} U(x,y)\omega_z^F(x,y,L)\,\mathrm{d}A - \int_{A_1} U(x,y)\omega_z^F(x,y,0)\,\mathrm{d}A\right) = SL\tau(\mathbf{u}^F). \end{aligned} \tag{43}$$

It is evident, in the case of displacement boundary condition (32)

$$\omega_z^F(x,y,0) = \Omega^F(0) = 0. \tag{44}$$

Results obtained above can be formulated in the next theorem.

Theorem 2. *In the case of torsion-free bending state the generalized twist*

$$\tau(\mathbf{u}^F) = \frac{\Omega^F(L) - \Omega^F(0)}{L} \tag{45}$$

vanishes, where $\Omega^F(z)$ is given by the equation

$$\Omega^F(z) = \frac{\int_A U(x,y)\omega_z^F(x,y,z)\,\mathrm{d}A}{\int_A U(x,y)\,\mathrm{d}A}. \tag{46}$$

This characterization of the torsion-free bending state is in harmony with Veubeke's result which formulates a property of shear centre based on Trefftz's definition (Veubeke, 1955).

4. Conclusions

In this paper, a new formula for the generalized twist is presented which holds for nonhomogeneous and anisotropic beams of solid cross section. The assumed form of the anisotropy is specified by Eq. (1). The material properties of the beam do not depend on the axial coordinate. Following Day, Podio-Guiduglio and Ieaşan (Day, 1981; Podio Guiduglio, 1983; Ieaşan, 1986, 1987) the concept of the generalized Saint-Venant's problem of torsion is introduced and its connection with the generalized twist is analysed. Expression of the generalized twist was given in terms of warping function of twisted cross section by Day, Podio-Guiduglio and Ieaşan (Day, 1981; Podio Guiduglio, 1983; Ieaşan, 1986, 1987).

Here, the generalized twist is expressed in terms of the axial component of the infinitesimal rotation vector weighted by the Prandtl's stress function of Saint-Venant's torsional problem.

In Section 3, a characterization of the torsion-free bending problem is presented by the use of generalized twist. The statement formulated in Theorem 2 accords with Veubeke's result on a characteristic property of shear centre.

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